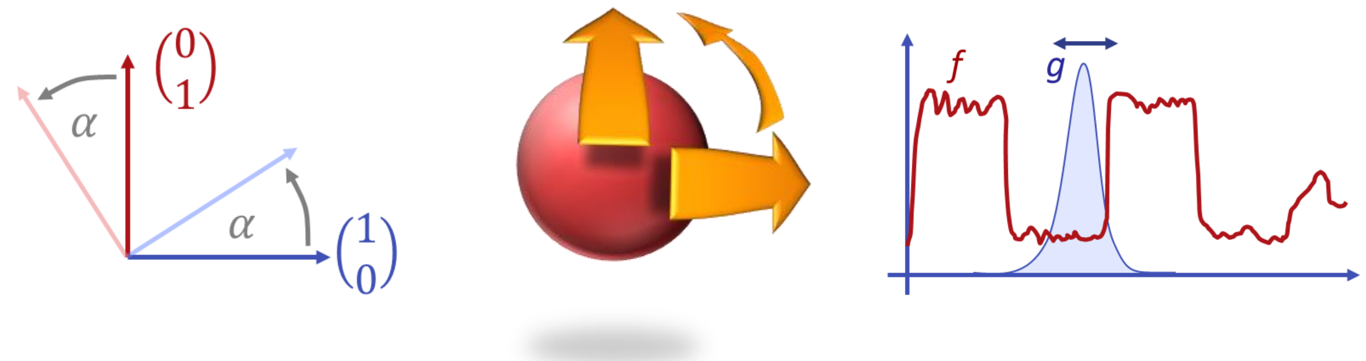
Modelling 1

SUMMER TERM 2020



LECTURE 3

Linear Mappings

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Linear Maps

Linear Maps

A function

- $f: V \to W$ between vector spaces V, W

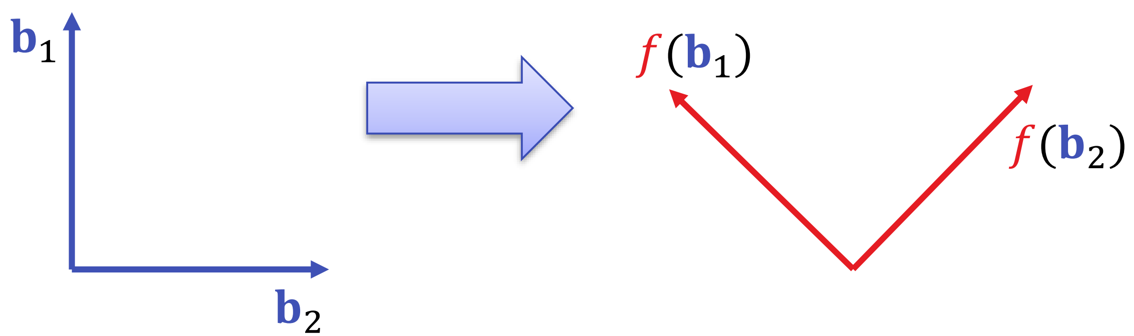
is linear if and only if:

- $\forall \mathbf{v}_1, \mathbf{v}_2 \in V$: $f(\mathbf{v}_1 + \mathbf{v}_2) = f(\mathbf{v}_1) + f(\mathbf{v}_2)$
- $\forall \mathbf{v} \in V, \lambda \in \mathbb{R}$: $f(\lambda \mathbf{v}) = \lambda f(\mathbf{v})$

Linear Maps

Constructing linear mappings:

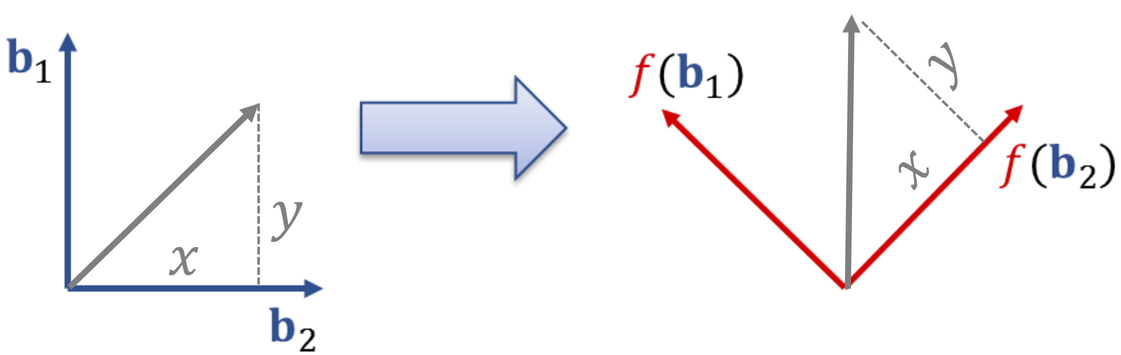
A linear map is uniquely determined if we specify a mapping value for each basis vector of V.



Matrix Representation

Finite dimensional spaces

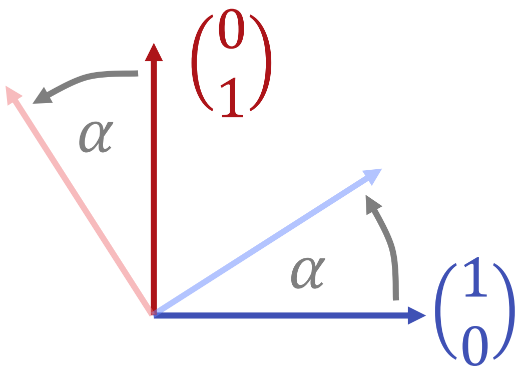
- Linear maps can be represented as matrices
  - For each basis vector $\mathbf{b}_i$:
    specify the mapped vector $\mathbf{a}_i$
  - Write in columns



$$f(x, y) = x \cdot f(\mathbf{b}_1) + y \cdot f(\mathbf{b}_2)$$

Columns = Images of Basis Vectors

Example: rotation matrix



$$\mathbf{M}_{rot} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

Linear Maps

Purely linear polynomial in coordinates of x:

$$f : \mathbb{R}^2 \to \mathbb{R}^3$$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to f(\mathbf{x}) = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \\ a_{31}x_1 + a_{32}x_2 \end{pmatrix}$$

$$f(\mathbf{x}) = \begin{pmatrix} x_1^2 \\ x_1 x_2 \\ \sin x_1 + x_1/x_2 \\ x_1 + 1 \end{pmatrix}$$

Linear Maps

Affine Maps:

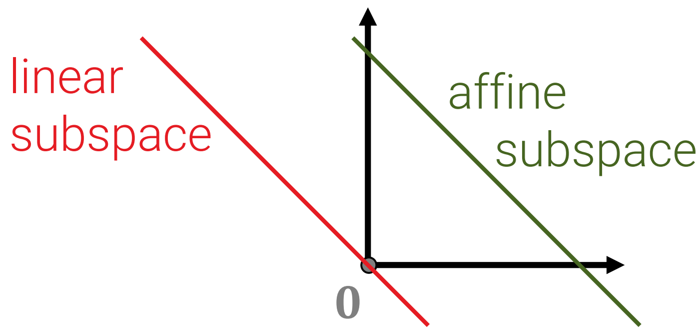
- Linear + constant function

$$f: \mathbb{R}^2 \to \mathbb{R}^3$$

$$f(\mathbf{x}) = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + t_1 \\ a_{21}x_1 + a_{22}x_2 + t_2 \\ a_{31}x_1 + a_{32}x_2 + t_3 \end{pmatrix}$$

$$= \mathbf{A} \cdot \mathbf{x} + \mathbf{t}$$

Affine Subspaces



Linear Subspace:

- Line / plane / hyperplane through origin

Affine Subspace

- Line / plane / hyperplane anywhere
- “*affine*” = “*linear* + *translation”* (adding constant)

Combinations of Linear Maps

Concatenation of linear maps are linear:

- Linear maps

$$f: V_1 \to V_2$$

$$g: V_2 \to V_3$$

- Concatenation

$$f \circ g: V_1 \to V_3$$

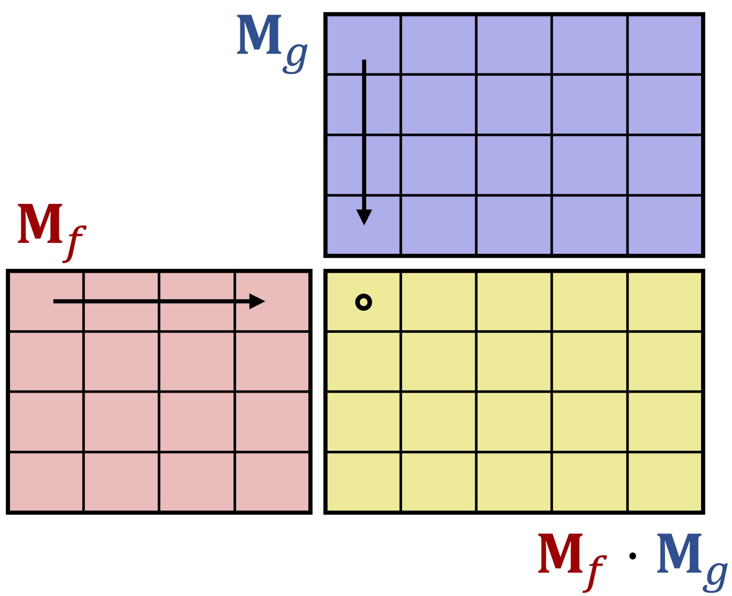
$$f \circ g(x) = f\big(g(x)\big)$$

  - $f \circ g$ is a linear again (easy to prove).
  - Linear mappings are closed w.r.t. to "$\circ$"
- Same holds for affine maps.

Matrix Multiplication

Composition of linear maps corresponds to matrix products:

- $f(g) = f \circ g = \mathbf{M}_f \cdot \mathbf{M}_g$
- Matrix product calculation:



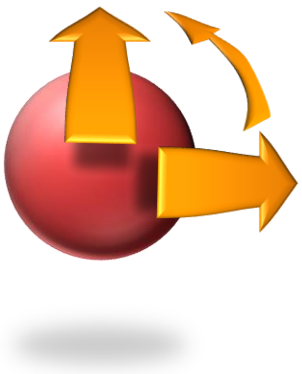
The (i, j)-th entry is the dot product of row i of $\mathbf{M}_f$ and column j of $\mathbf{M}_g$

Algebraic Structure of Linear Maps

General Linear Group GL(n)

Relevant example:

- Invertible $d \times d$ square matrices $\mathrm{GL}(d) = \left(\mathbb{R}^{d \times d}, \ \cdot\ \right)$
- Subgroups:
  - orthogonal group:
    $d \times d$ rotation & reflection matrices $\mathrm{O}(d) \subset GL(d)$
  - special orthogonal group (rotation group):
    $d \times d$ rotation matrices $\mathrm{SO}(d) \subset O(d)$
- None are commutative for $d > 1$



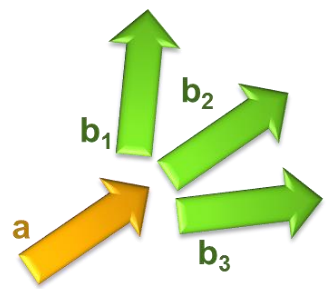
Notation

Affine mappings

- Rigid motions $SE(d)$ (special Euclidean group):
  - All combinations of $SO(d)$ and translations
  - Rotations & translations
- Rigid motions $E(d)$ (Euclidean group):
  - All combinations of $O(d)$ and translations
  - Rotations, reflections & translations
- Representation

$$f(\mathbf{x}) = \mathbf{A} \cdot \mathbf{x} + \mathbf{t}$$

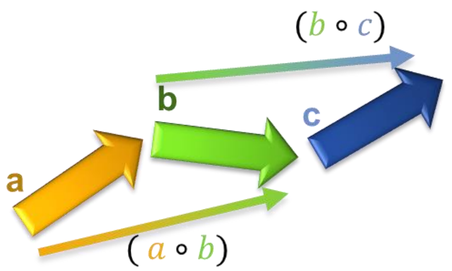
Group Structure



closed operation

all operations always possible

$\forall a, b \in G\colon\ a \circ b \in G$



associativity

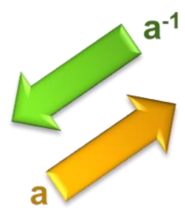
effect "adds up"

$\forall a, b, c \in G\colon (\ a \circ b) \circ c = a \circ (b \circ c)$

Neutral element

unique null operation

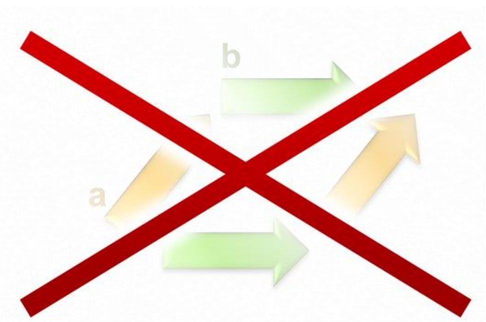
$\forall a \in G\colon\ a \circ id = a$



Inverse

all operations reversible

$\forall a \in G\colon\ a \circ a^{-1} = id$



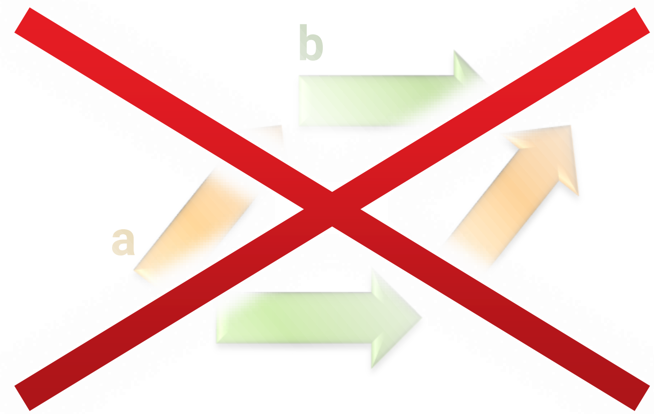
not commutative

commutativity

intuition: flat structure

$\forall a, b \in G\colon\ a \circ b = b \circ a$

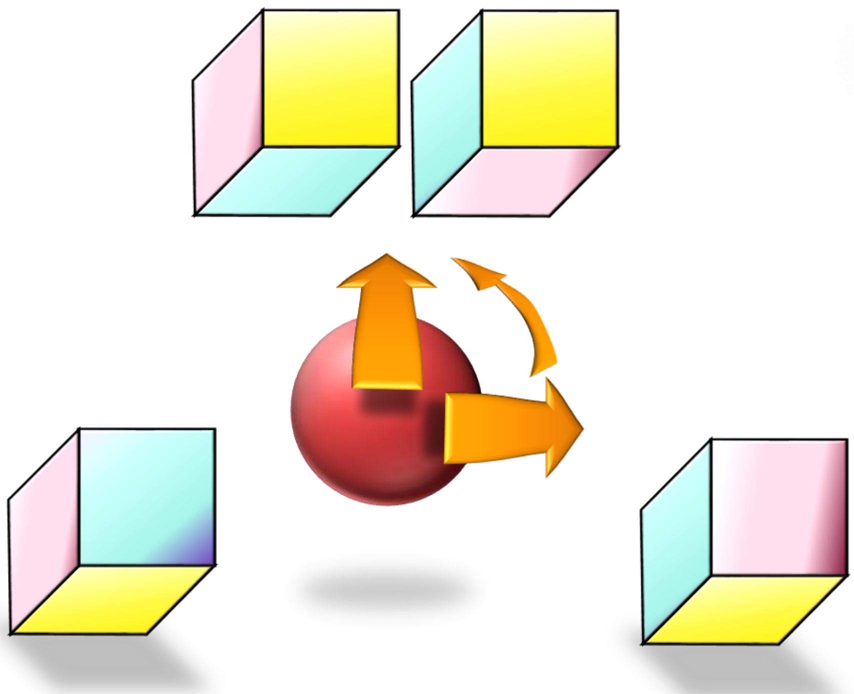
Not Commutative!

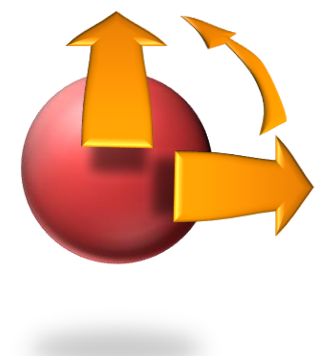


commutativity

intuition: flat structure

$\forall a, b \in G: a \circ b = b \circ a$





Matrix Algebra

Matrix Algebra

Define three operations

- Matrix addition

$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{bmatrix} + \begin{bmatrix} b_{1,1} & \cdots & b_{1,n} \\ \vdots & \ddots & \vdots \\ b_{m,1} & \cdots & b_{m,n} \end{bmatrix} = \begin{bmatrix} a_{1,1} + b_{1,1} & \cdots & a_{1,n} + b_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} + b_{m,1} & \cdots & a_{m,n} + b_{m,n} \end{bmatrix}$$

- Scalar matrix multiplication

$$\lambda \cdot \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{bmatrix} = \begin{bmatrix} \lambda \cdot a_{1,1} & \cdots & \lambda \cdot a_{1,n} \\ \vdots & \ddots & \vdots \\ \lambda \cdot a_{m,1} & \cdots & \lambda \cdot a_{m,n} \end{bmatrix}$$

- Matrix-matrix multiplication

$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{bmatrix} \cdot \begin{bmatrix} b_{1,1} & \cdots & b_{1,m} \\ \vdots & \ddots & \vdots \\ b_{k,1} & \cdots & b_{k,m} \end{bmatrix} = \begin{bmatrix} \ddots & & ⋰ \\ & \sum_{q=1}^{k} a_{q,j} \cdot b_{i,q} & \\ ⋰ & & \ddots \end{bmatrix}$$

Algebraic Rules: Addition

Addition: like real numbers ("commutative group")

- Prerequisites:
  - Number of rows match
  - Number of columns match
- Associative: $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$
- Commutative: $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
- Subtraction: $\mathbf{A} + (-\mathbf{A}) = \mathbf{0}$
- Neutral Op.: $\mathbf{A} + \mathbf{0} = \mathbf{A}$

Settings

$\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{R}^{n \times m}$

(matrices, same size)

Alg. Rules: Scalar Multiplication

Scalar Multiplication: Vector space

- Prerequisites:
  - Always possible
- Repeated Scaling: $\lambda(\mu \mathbf{A}) = \lambda\mu(\mathbf{A})$
- Neutral Operation: $1 \cdot \mathbf{A} = \mathbf{A}$
- Distributivity 1: $\lambda(\mathbf{A} + \mathbf{B}) = \lambda\mathbf{A} + \lambda\mathbf{B}$
- Distributivity 2: $(\lambda + \mu)\mathbf{A} = \lambda\mathbf{A} + \mu\mathbf{A}$

Settings
$\lambda \in \mathbb{R}$
$\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times m}$
(same size)

So far:

- Matrices form vector space
- Just different notation, same semantics!

Algebraic Rules: Multiplication

Multiplication: Non-Commutative Ring / Group

- Prerequisites:
  - Number of columns right = number of rows left
- Associative: $(\mathbf{A} \cdot \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot (\mathbf{B} \cdot \mathbf{C})$
- Not commutative: often $\mathbf{A} \cdot \mathbf{B} \neq \mathbf{B} \cdot \mathbf{A}$
- Neutral Op.: $\mathbf{A} \cdot \mathbf{I} = \mathbf{A}$

- Inverse: $\mathbf{A} \cdot (\mathbf{A}^{-1}) = \mathbf{I}$
  - Additional prerequisite:
    - Matrix must be square!
    - Matrix must have full rank

Subset of invertible matrices only:

$GL(d) \subset \mathbb{R}^{d \times d}$

"general linear group"

Algebraic Rules: Multiplication

Multiplication: Non-Commutative Ring

Settings

$\mathbf{A} \in \mathbb{R}^{n \times m}$

$\mathbf{B} \in \mathbb{R}^{m \times k}$

$\mathbf{C} \in \mathbb{R}^{k \times l}$

- Prerequisites:
  - Number of columns right = number of rows left
- Associative: $(\mathbf{A} \cdot \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot (\mathbf{B} \cdot \mathbf{C})$
- Not commutative: often $\mathbf{A} \cdot \mathbf{B} \neq \mathbf{B} \cdot \mathbf{A}$
- Neutral Op.: $\mathbf{A} \cdot \mathbf{I} = \mathbf{A}$

- Inverse: $\mathbf{A} \cdot (\mathbf{A}^{-1}) = \mathbf{I}$
  - Additional prerequisite:
    - Matrix must be square!
    - Matrix must have full rank

Subset of invertible matrices only:

$GL(d) \subset \mathbb{R}^{d \times d}$

"general linear group"

Transposition Rules

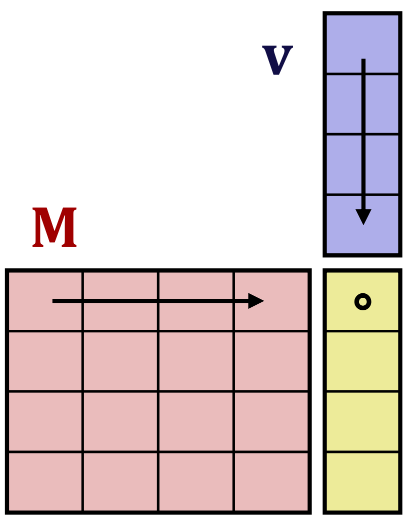
Transposition

- Addition: $(\mathbf{A}+\mathbf{B})^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}} + \mathbf{B}^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}} + \mathbf{A}^{\mathrm{T}}$
- Scalar-mult.: $(\lambda\mathbf{A})^{\mathrm{T}} = \lambda\mathbf{A}^{\mathrm{T}}$
- Multiplication: $(\mathbf{A}\cdot\mathbf{B})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\cdot\mathbf{A}^{\mathrm{T}}$
- Self-inverse: $\left(\mathbf{A}^{\mathrm{T}}\right)^{\mathrm{T}} = \mathbf{A}$
- (Inversion:) $(\mathbf{A}\cdot\mathbf{B})^{-1} = \mathbf{B}^{-1}\cdot\mathbf{A}^{-1}$
- Inverse-transp.: $\left(\mathbf{A}^{\mathrm{T}}\right)^{-1} = (\mathbf{A}^{-1})^{\mathrm{T}}$
- Othogonality: $[\mathbf{A}^{\mathrm{T}} = \mathbf{A}^{-1}] \Leftrightarrow [\mathbf{A}\ \textbf{is orthogonal}]$

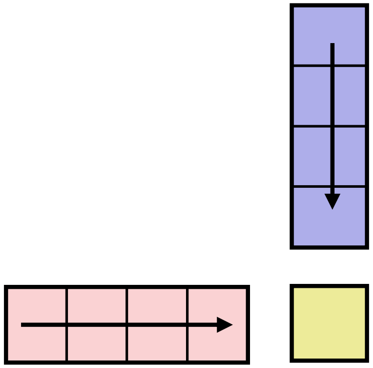
General Matrix Product (Notation)

All operations are matrix-matrix products:

- Matrix-Vector product:
- $f(\mathbf{x}) = \mathbf{M}_f \cdot \mathbf{x}$



Vectors



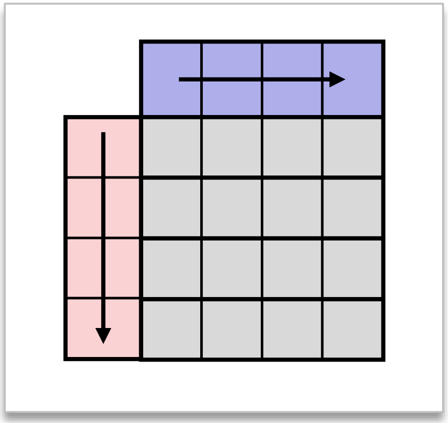
$$\mathbf{x}^{\mathrm{T}} \cdot \mathbf{y} \rightarrow \mathbb{R}$$

Inner product

- Matrix-product **row** · **column**

$$„\mathbf{x} \cdot \mathbf{y}“ = \langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^{\mathrm{T}} \cdot \mathbf{y}$$

New: Outer Product



$$\mathbf{x} \in \mathbb{R}^n$$
$$\mathbf{y} \in \mathbb{R}^m$$
$$\mathbf{x} \cdot \mathbf{y}^{\mathrm{T}} \rightarrow \mathbb{R}^{n \times m}$$

Outer product

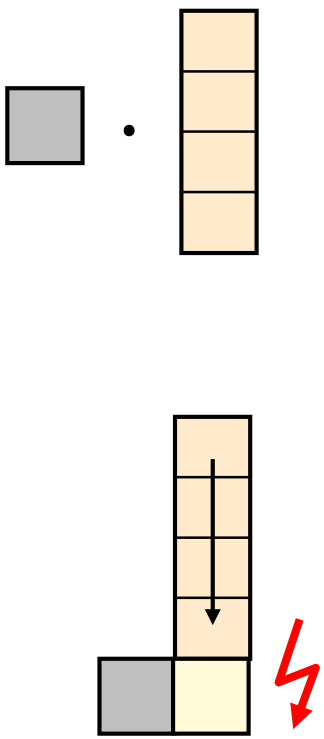
- Matrix-product **column** · **row**

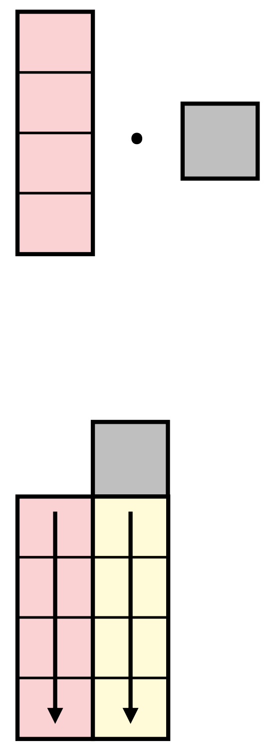
  $\mathbf{x} \cdot \mathbf{y}^{\mathrm{T}}$
- Yields a matrix (rank ≤ 1)
- We'll need this later…

Scalar Product

NOT OK

OK





Scalar Product

Matrix Algebra:

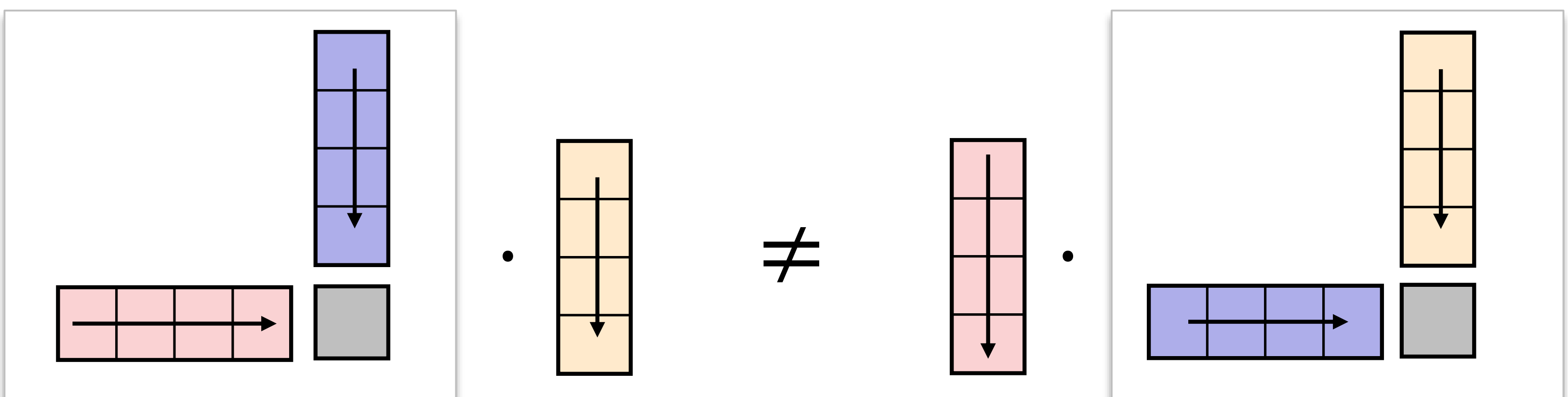
- Scalar product is a special case

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^{\mathrm{T}} \cdot \mathbf{y}$$

- Caution when mixing with scalar-vector product!

$$\langle \mathbf{x}, \mathbf{y} \rangle \cdot \mathbf{z} \neq \mathbf{x} \cdot \langle \mathbf{y}, \mathbf{z} \rangle$$
$$\left(\mathbf{x}^{\mathrm{T}} \cdot \mathbf{y}\right) \cdot \mathbf{z} \neq \mathbf{x} \cdot \left(\mathbf{y}^{\mathrm{T}} \cdot \mathbf{z}\right)$$

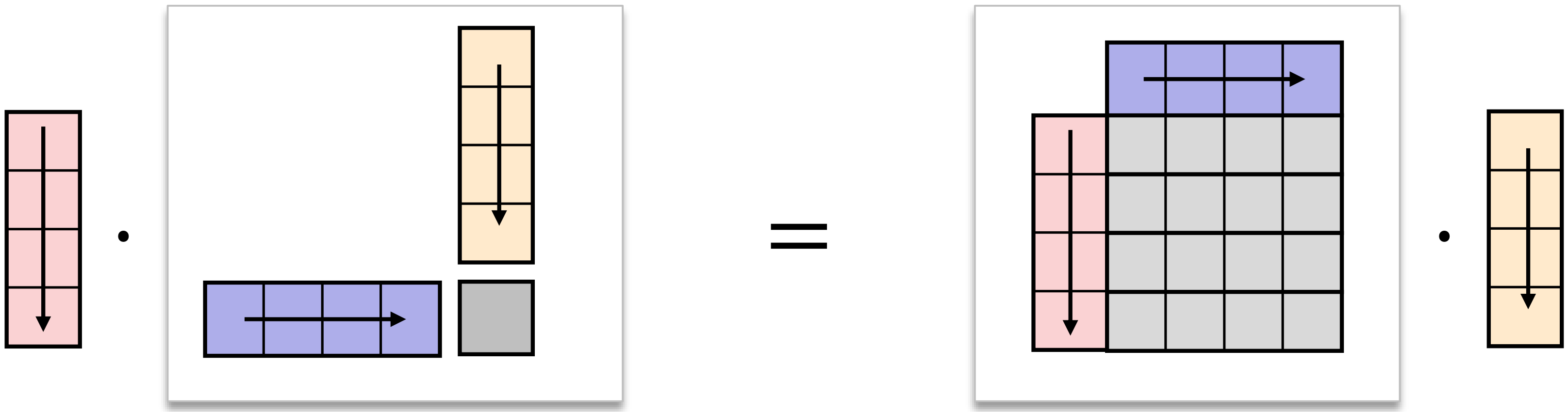
Scalar multiplication not a matrix-product!



Matrix Algebra Example

Associativity with outer product

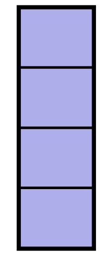
$$\mathbf{x} \cdot \langle \mathbf{y}, \mathbf{z} \rangle = \mathbf{x} \cdot \left(\mathbf{y}^{\mathrm{T}} \cdot \mathbf{z}\right) = \left(\mathbf{x} \cdot \mathbf{y}^{\mathrm{T}}\right) \cdot \mathbf{z}$$



Vectors

Vectors

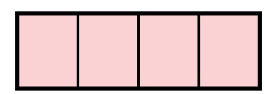
- Column matrices
- Matrix-Vector product consistent



$\mathbf{x} \in \mathbb{R}^d$

Co-Vectors

- “projectors”, “dual vectors”, “linear forms”, “row vectors”
- Vectors to be projected on



$\mathbf{y}^\mathrm{T} \in \mathbb{R}^d$

Transposition

- Convert vectors into projectors and vice versa